ON TAX POLICY IN A DYNAMIC REAL TRADE MODEL*

Marcelo Bianconi**

ABSTRACT

This paper studies the steady state and dynamic implications of differences in tax policies and/or discount factors in an overlapping generations two-country two-good dynamic model of international trade. It is shown that in the overlapping generations framework, production diversity along with factor price equalization obtains in the long run. Contrary to conventional wisdom, it is shown that countries with identical tax policies may attain to very distinct steady state capital stocks. The nonlinear maps generated by the structural equilibrium induce multiple steady state equilibrium or endogenous cycles depending on the relative factor intensities in the production of the two goods.

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** Assistant Professor of Economics, Department of Economics, 305 Braker Hall, Tufts University, Medford, MA 02155. Ph. (617) 6285000-2677; Fax (617) 6273917.

I. INTRODUCTION

Recently, economists have leaned towards modeling the economic phenomena from a dynamic, as opposed to a static, perspective. The analysis I carry out below is an attempt to integrate the traditional static trade theoretic framework, represented by the two-sector/two-good model of production, with a dynamic life-cycle saving model. Early attempts to analyze trade theory from a dynamic perspective include Oniki and Uzawa(1965) and Stiglitz(1970) among others. The contribution of this paper with respect to those authors is to fully specify the saving-investment demand side decision according to the life-cycle paradigm.[1]

In particular, my work draws on the recent contributions by Galor and Lin(1989) and Galor(1991), but the subject of analysis is quite distinct. Those authors were concerned with cases where convergence to a unique steady state equilibrium was assured whereas I focus on cases of multiple and cyclical equilibria. It also relates closely to the recent contribution of Baxter(1992) where the demand side is modeled according to the infinitely lived representative agent paradigm. It is established here that in the life-

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1 *More recently, Matsuyama(1988a,b) has studied dynamic trade models with an explicit life-cycle saving-investment decision. However, in the two papers above, he restricts the analysis to the case of a small open economy. See also Eaton(1987). Baxter(1988) advocates the use of the 2×2×2 model of international trade with an infinitely lived maximizing model of the consumer as a tool for open economy macroeconomic issues, however her paper only deals with the small open economy; Baxter(1992) is an extension which introduces fiscal policy and a two-country world. My analysis below is generalized to autarky, small open economy, and two-country world. In particular, I show an example of a small open economy where no dynamics arise.*

cycle saving model long run production diversity obtains whereas in Baxter(1992)[and Stiglitz(1970)] long run specialization obtains.

The paper fits in several other strands of the literature. First, I am going to analyze the model from the point of view of the issue of cross-country disparities in economic growth. The standard neoclassical growth model generates a convergence hypothesis that is at odds with the real world evidence.[2] Lucas(1988) [section 4] has put forth a view that economies with different initial conditions will display no tendency to converge, a view also shared by Boldrin and Scheinkman(1988). More recently, Parente and Prescott(1991) put forth a view that economies with different effective tax rates will display no tendency to converge and they report results on level differences due to alternative tax rates. Authors such as Jones and Manuelli(1990) and Rebelo(1991), who use alternative endogenous growth models, also point out to tax policy as one of the determinants of cross-country differences in economic growth. Using a two-sector life-cycle model, I construct some examples below that lend support to the former view. Specifically, using nonlinear dynamic methods I show that countries with identical interest income tax rates and/or discount factors may diverge (in

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2 *Quah(1989) documents and interprets the econometric evidence on the persistence of cross-country disparities in economic growth and Romer(1989) extensively surveys the recent literature. See also Benhabib and Jovanovic(1991) for an account of growth disparities that does not rely on increasing returns or externalities, but on the persistence of the Solow residual.*

levels), and the determinant of the long run equilibrium will be its initial condition.[3]

Second, the paper is intended to provide an extension of Diamond(1965) for the case of non-joint production in two distinct sectors: a consumption good sector and a capital good sector. In this first attempt, I abstract from issues of government debt and international borrowing and lending. I introduce an interest income tax rate with incidence on the interest received by the retired old generation coupled with a lump-sum transfer scheme to the same retired old generation. By treating the interest income tax rate and the discount factor as parameters, I show that for certain values of the specific parameter the economy may exhibit multiple steady state equilibria or endogenous cyclical behavior under perfect foresight.

Third, I start to integrate the Diamond(1965) model with the Oniki and Uzawa(1965) model of dynamic international trade along the lines of Galor and Lin(1989). I obtain a two-country tractable dynamic real trade model which appears to be suitable for the analysis of a variety of economic questions, one of which is tax policy. More recently, Sibert(1990) has analyzed issues of capital taxation in a two-country one-sector overlapping generations model with borrowing and lending, based on Diamond(1965). My paper may be seen as an extension of that work to a two-sector production setting, but it restricts the analysis to the case of no borrowing and lending. However, one of the main interests here, as opposed to the comparative statics perspective of

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3 *Note that in my model the nonlinear maps induce either multiple equilibria or endogenous cycles; it is not an endogenous growth model.*

Sibert(1990), is to consider tax policy from the Parente and Prescott(1991) perspective, i.e. different countries have different tax rates and this may be a factor explaining cross-country economic disparities.

Fourth, the analysis here takes advantage of the fact that the structural equilibrium model that emerges consists of a set of nonlinear maps. I use an available tool in nonlinear dynamics, bifurcation theory, to analyze the possibility of multiple equilibria and cyclical behavior in the model. In doing so, it extends the work of Boldrin(1989) and Boldrin and Deckenere(1990) to the case of life-cycle saving, and integrates the two-sector life-cycle model to the growing existing literature on nonlinear dynamic economic models extensively surveyed by Boldrin and Woodford(1990). For instance, Reichlin(1987) is an attempt in this direction but, he uses fixed proportions technology in both sectors whereas here I assume constant returns to scale technology in both sectors.[4]

The paper is organized as follows: section II presents the economic model; section III solves for the dynamic equilibrium; section IV analyzes the steady state equilibrium and pattern of trade, the steady state production possibilities frontier, and the issue of efficiency; section V examines the local structural stability, dynamic behavior, multiplicity of equilibria, and cyclical behavior of the autarky, small open, and two-country economies;

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4 *More recently, Matsuyama(1991) has analyzed a model of industrialization with increasing returns from the perspective of bifurcation theory emphasizing the role of government policy and agricultural productivity. See also the readings in Benhabib(1992).*

section VI concludes while a few proofs and a description of an important result are left to an Appendix.

II. THE MODEL

Consider a competitive world consisting of two countries, D and F, each producing a pure consumption good, X_t, and a pure capital good, Y_t, traded over every period t. The two goods are produced using labor, $L_t^{i,j}$, and capital, $K_t^{i,j}$, for i=X,Y; and j=D,F. The labor endowment in each country is fixed. The countries are engaged in free trade in goods. There is neither international labor mobility nor international borrowing and lending. Without loss of generality, capital is assumed to depreciate fully each period. Perfect foresight is assumed throughout.

Technology in each country[5], in each sector, consists of time invariant constant returns to scale production functions which imply

$$X_t^j = L_t^{x,j} f_x(k_t^{x,j}) \tag{1a}$$

$$Y_t^j = L_t^{y,j} f_y(k_t^{y,j}) \tag{1b}$$

where $k_t^{i,j}=(K_t^{i,j}/L_t^{i,j})$ is the capital-labor ratio in sector i and country j, and $f_i(.)$ is the production function in sector i, common to both countries, which is twice continuously differentiable, positive, increasing, and strictly concave, or

$$f_i(k^{i,j})>0; \qquad f'_i(k^{i,j})>0; \qquad f''_i(k^{i,j})<0 \tag{1c}$$

also f(.) satisfies the usual Inada conditions. In each country, the relative factor intensities may be simply summarized: if $k^{x,j}>k^{y,j}$ then the consumption

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5 *The production side of the model is based on the two-sector model of growth as in Uzawa(1961, 1963).*

good is more capital intensive than the capital good; if $k^{x,j}<k^{y,j}$ then the capital good is more capital intensive than the consumption good. Factor intensity reversals are ruled out throughout the paper, by assumption. In each country, both goods are produced with K and L perfectly mobile across sectors. If both goods are produced, the zero profit conditions for firms yield

$$r_t^j = p_t^j\ f'_x(k_t^{x,j}) = f'_y(k_t^{y,j}) \tag{2a}$$

$$w_t^j = p_t^j[f_x(k_t^{x,j}) - f'_x(k_t^{x,j})k_t^{x,j}] = [f_y(k_t^{y,j}) - f'_y(k_t^{y,j})k_t^{y,j}] \tag{2b}$$

where r is the rental rate, w is the wage rate, and p is the price of the consumption good in terms of the price of the capital good, the latter normalized to unity, i.e. the capital good is assumed to be the numeraire.

It is straightforward to show that $k_t^{i,j} = k^{i,j}(\Omega_t^j)$, where Ω_t^j is the wage-rental ratio and k(.) is strictly decreasing. From (2a), if both goods are produced (I assume this to be the case throughout the paper) one then obtains

$$p_t^j(\Omega_t^j) = f'_y[k^{y,j}(\Omega_t^j)] \ /\ f'_x[k^{x,j}(\Omega_t^j)] \tag{3}$$

which yields $\Omega_t^j = \Omega(p_t^j)$, with $\Omega(.)$ strictly increasing. In turn, the rental and wage rates are uniquely determined as a function of the relative price

$$w_t^j = w(p_t^j) \tag{4a}$$

$$r_t^j = r(p_t^j). \tag{4b}$$

Finally, the per worker production of each good, in each country, is given by

$$X_t^j/L^j = x_t^j = [(k_t^j-k_t^{y,j})/(k_t^{x,j}-k_t^{y,j})]f_x[k^{x,j}(\Omega(p_t^j))] = x(p_t^j,k_t^j) \tag{5a}$$

$$Y_t^j/L^j = y_t^j = [(k_t^{x,j}-k_t^j)/(k_t^{x,j}-k_t^{y,j})]f_y[k^{y,j}(\Omega(p_t^j))] = y(p_t^j,k_t^j) \tag{5b}$$

where L^j is aggregate labor in country j, and $k_t^j=K_t^j/L^j$ is the aggregate capital-labor ratio in country j.

The demand side consists of life cycle consumption-saving behavior. Every period t, L^j individuals are born in country j. Identical individuals live for two periods. In the first period, the young generation works at the competitive wage w_t^j, and allocates its human wealth between consumption and saving. In the second period, the now old generation retires consuming all after-tax savings plus a transfer from the government. The problem is

$$\max_{\{c_{1t}^j, s_t^j\}} \quad u(c_{1t}^j) + \beta^j\, u(c_{2t+1}^j) \tag{6}$$

subject to

$$s_t^j = w_t^j - p_t^j\, c_{1t}^j \tag{6a}$$

$$p_{t+1}^j\, c_{2t+1}^j = r_{t+1}^j(1-\tau^j)\, s_t^j + T_{t+1}^j \tag{6b}$$

where $0<\beta<1$ is the discount factor, c_1 is per labor consumption when young, c_2 is per labor consumption when old, s is saving, $0<\tau<1$ is the tax rate on interest income, and T is the per labor lump-sum transfer.[6] The function u(.) is a twice continuously differentiable, monotonically increasing, and quasi-concave utility function, or $u'(.)>0$, $u''(.)<0$, satisfying the usual Inada conditions. Solution of (6) implies a smooth savings function of the form

$$s_t^j = s(w_t^j,\ r_{t+1}^j,\ p_{t+1}^j,\ p_t^j,\ T_{t+1}^j;\ \tau^j,\ \beta^j) \tag{7}$$

where $\partial s/\partial w_t^j>0$ under the assumption that consumption in both periods is a normal good, $\partial s/\partial \tau^j<0$, and $\partial s/\partial \beta^j<0$. I shall assume throughout the paper that $\partial s/\partial r_{t+1}^j \geq 0$. If $\partial s/\partial r_{t+1}^j>0$, it implies that the substitution effect dominates, or alternatively that the elasticity of substitution between

6 *Diamond(1970) considers a similar tax scheme in a one-sector overlapping generations model of a closed economy. Another possibility would be to transfer in a lump-sum fashion to the young generation, see e.g. Atkinson and Stiglitz(1980), Lecture 8. I chose the Diamond(1970) scheme because it conveniently captures the distortionary effect of taxes.*

consumption in the two periods is greater than one. If $\partial s/\partial r_{t+1}{}^{j}=0$, it implies that the elasticity of substitution is equal to one, or alternatively the logarithmic utility function.

The government, in each country, is assumed to transfer all receipts back to individuals according to the rule

$$T_t{}^j = r_t{}^j \tau^j s_{t-1}{}^j. \tag{8}$$

III. DYNAMIC TRADE EQUILIBRIUM

Equations (4) and (8) may be substituted into (7) yielding the equilibrium saving function. If $\partial s/\partial r_{t+1}{}^{j}>0$, then

$$s_t{}^j=s(.)=S(p_t{}^j,p_{t+1}{}^j;\tau^j,\beta^j)$$

where $\partial S/\partial p_t{}^j = [(\partial s/\partial w)(\partial w/\partial p_t{}^j)+(\partial s/\partial p_t{}^j)]$,

$\partial S/\partial p_{t+1}{}^j = \{-p_t[(\partial c_1/\partial p_{t+1}{}^j)+(\partial c_1/\partial r)(\partial r/\partial p_{t+1}{}^j)]\}$,

$\partial S/\partial \tau^j<0$ and $\partial S/\partial \beta^j<0$.

If $k^x>k^y$ and $\partial s/\partial r_{t+1}{}^j>0$, then $\partial S/\partial p_{t+1}{}^j>0$ and the sign of $\partial S/\partial p_t{}^j$ is ambiguous. If $k^x<k^y$ and $\partial s/\partial r_{t+1}{}^j>0$, then $\partial S/\partial p_{t+1}{}^j<0$ and $\partial S/\partial p_t{}^j>0$.

In equilibrium, investment equals saving in each country, or

$$k_{t+1}{}^j = S(p_t{}^j,p_{t+1}{}^j;\tau^j,\beta^j). \tag{9a}$$

The goods market equilibrium in the jth country capital good sector requires, by (5b), that

$$S(p_t{}^j,p_{t+1}{}^j;\tau^j,\beta^j) = y(p_t{}^j,k_t{}^j). \tag{9b}$$

The world dynamic equilibrium is then obtained by considering each country saving/investment equilibrium(since international borrowing and lending does not exist) and the equilibrium condition in the world capital good sector, or

$$k_{t+1}{}^D = S(p_t,p_{t+1};\tau^D,\beta^D) \tag{10a}$$

$$k_{t+1}{}^{F} = S(p_t, p_{t+1}; \tau^F, \beta^F) \qquad (10b)$$

$$S(p_t, p_{t+1}; \tau^D, \beta^D) + S(p_t, p_{t+1}; \tau^F, \beta^F) = y(p_t, k_t{}^D) + y(p_t, k_t{}^F) \qquad (10c)$$

with $k_o{}^D$ and $k_o{}^F$ exogenously given.[7]

IV. STEADY STATE EQUILIBRIUM AND PATTERN OF TRADE

The steady state trade equilibrium consists of the fixed point $\{k^D, k^F, p\}$ which satisfy

$$k^D = S(p,p;\tau^D,\beta^D) \qquad (11a)$$

$$k^F = S(p,p;\tau^F,\beta^F) \qquad (11b)$$

$$S(p,p;\tau^D,\beta^D) + S(p,p;\tau^F,\beta^F) = y(p,k^D) + y(p,k^F). \qquad (11c)$$

It is clear that, under the implicit assumption that both goods are produced, the economies above satisfy the Stolper-Samuelson and the Rybczynski theorems. Also, if the two countries are equal in every respect, the trade equilibrium coincides with the autarky equilibrium [by inspection of (9)-(10)]. The steady state trade equilibrium satisfies the following propositions:[8]

<u>Proposition 1</u>: In the two country steady state equilibrium, assume: i. $k^{x,j} > k^{y,j}$, or the consumption good is capital intensive; ii. $\tau^D > \tau^F$, or the tax rate in the domestic country is greater than in the foreign country; iii. countries are identical in all other respects. Then, the low tax (foreign)

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7 *The equilibrium in every period requires that: i. production of the capital good equals aggregate saving; ii. production of the consumption good equals aggregate consumption; iii. supply and demand for capital and labor are equal in both sectors.*

8 *Proofs of Propositions 1, 2, and 3 are given in the Appendix.*

country exports the capital intensive good while the high tax (domestic) country exports the labor intensive good.

Intuitively the proposition above is simple because the high tax rate on interest income decreases saving, and the low saving implies a low capital stock in the high tax country, relative to the low tax country. This leads to a shift, in the high tax country, to the production of the labor intensive good while the opposite occurs in the low tax country.

<u>Proposition 2</u>: In the autarky steady state equilibrium, assume: i. $k^{x,j} > k^{y,j}$, or the consumption good is capital intensive; ii. $\tau^D > \tau^F$, or the tax rate in the domestic country is greater than in the foreign country; iii. countries are identical in all other respects. Then, if trade opens up, in the two country steady state equilibrium: i. the wage rate decreases and the rental rate increases in the low tax (foreign) country; ii. the wage rate increases and the rental rate decreases in the high tax (domestic) country.

The intuition behind this proposition follows from the previous one. If trade opens up, the low tax country shall demand more capital therefore pushing up its rental rate. In the high tax country, the demand for labor increases pushing up the wage rate.

<u>Proposition 3</u>: In the two country steady state equilibrium, assume: i. $k^{x,j} > k^{y,j}$, or the consumption good is capital intensive; ii. $\tau^D \neq \tau^F$, or the tax rate in the domestic country is different from that in the foreign country; iii. countries are identical in all other respects; iv. both goods are produced in each country. Trade equalizes factor prices in both countries.

The result of Proposition 3 illustrates one of the basic features of the two-sector life-cycle framework. In the infinite horizon single agent model, of Stiglitz(1970) say, trade leads to long-run specialization of at least one of the countries inhibiting long-run factor price equalization. In the two-sector life-cycle economy, diversification in production in both countries is feasible and long-run factor price equalization holds. Note that propositions 1, 2, and 3 also hold for the case when $\tau^D = \tau^F$, but $\beta^D \neq \beta^F$, see e.g. Galor and Lin(1989).

Propositions 1-3 establish the result that differences in tax rates on interest income (or in the discount factor) determine the pattern of trade in the long run. The result is conceptually consistent with the traditional Hecksher-Ohlin theory of trade, even though that theory emphasizes differences in factor endowments. In this sense, tax policy defined according to Parente and Prescott(1991) may be one of the determinants of the pattern of trade in the long run.

IV.1. The Steady State Production Possibilities Frontier(PPF)

Assume, without loss of generality, a one country world under autarky. The steady state PPF may be conveniently summarized by the expression, see e.g. Jones(1965),

$$L = L^x+L^y = (L^x/X)\ X + (L^y/Y)\ Y.$$

Its slope follows directly as

$$dY/dX|_{L^x+L^y=L} = -\ f_y(k^y)/f_x(k^x).$$

From above it is easy to show that the steady state PPF is nonlinear. From the first order conditions of problem (6), we have that, in steady state,

$$(1-\tau)f_y'(k^y)=[u'(c_1)/\beta u'(c_2)].$$

This implies that the solution for the capital/labor ratio in the capital good sector, k^y, is not independent of the consumer demands, or alternatively k^y is not independent of the composition of output between X and Y. In effect, as the composition between X and Y varies, the capital/labor ratios also vary and the steady state PPF is nonlinear.

In a recent paper, Baxter(1992) shows that in a two-sector model of growth with an infinitely lived representative agent, the steady state PPF is linear implying that at least one country must specialize in the production of at least one good. I have just shown above that this is not the case in the overlapping generations model. How can one account for this difference? The technical answer is that in the infinitely lived representative agent model, the solution for the steady state capital/labor ratio in the capital good sector, k^y, is independent of the consumer demands, and given by

$$(1-\tau)f_y'(k^y)=(1/\beta).$$

There is a version of the dynamic nonsubstitution theorem of Mirrlees(1969) at work here. Alternatively, k^y is independent of the composition of output between X and Y and the steady state PPF is linear.

Intuitively, what accounts for the difference is that in the overlapping generations model there are two distinct consumers that are able to trade with each other at any point in time. In effect, they are able to borrow and lend from each other and this is reflected in the dependence of k^y on

$[u'(c_1)/u'(c_2)]$ which is the marginal rate of substitution between consumption of young and old in steady state. The overlapping generations economy does not satisfy the dynamic nonsubstitution theorem. In the case of the infinitely lived representative agent, he/she is not able to borrow and lend at any point in time because, in equilibrium, net private debt must be zero. In other words, there is only one consumer which implies that the marginal rate of substitution must be one, i.e. the consumer perfectly smooths its consumption stream. Even though the infinitely lived representative consumer framework is a very useful setting for the analysis of a variety of intertemporal dynamic problems, it is not as suitable for the analysis of consumer issued debt. In this specific problem, the absence of consumer heterogeneity implies that the economy must specialize in the production of at least one good in the steady state. Introducing heterogeneous consumers allows for steady state diversification of production.

IV.2. A Detour: Efficiency of Steady State Equilibrium

One important issue that arises in the class of overlapping generations models regards efficiency, see e.g. Galor and Ryder(1991). Assume again, without loss of generality, a one country world under autarky. Also assume that the planner does not discount for different generations, i.e. it treats all generations alike In the two-sector framework above, the efficient steady state equilibrium may be characterized by a pair of capital/labor ratios, k^{x*} and k^{y*}, and per labor consumptions of young and old, c_1^* and c_2^*, that solve

$$\text{Max} \quad u(c_1) + \beta\, u(c_2) \qquad (6')$$

subject to

$$f_x(k^x) = c_1 + c_2 \qquad (6a')$$

$$f_y(k^y) = k^x + k^y. \qquad (6b')$$

An efficient equilibrium is then characterized by k^{x*}, k^{y*}, c_1^*, and c_2^* that satisfy

$$f'_y(k^{y*}) = 1 \qquad (6c)$$

$$u'(c_1^*) = \beta\, u'(c_2^*) \qquad (6d)$$

and the resource constraints (6a')-(6b'). The solution is recursive with (6c) solving for k^y, (6b') solving for k^x, and (6d) and (6a') solving for c_1 and c_2. Equation (6c) denotes the golden rule of accumulation for this model, the marginal physical product equates the rate of depreciation.

In the economies presented in (1)-(10), the solution for k^y is given by $f'_y(k^y)=r[p(\tau,\beta)]=[u'(c_1)/\beta u'(c_2)(1-\tau)]$. If $k^y>k^{y*}$, or alternatively $f'_y(k^y)=[u'(c_1)/\beta u'(c_2)(1-\tau)]<1$, then the economy is overinvesting relative to the golden rule and the steady state equilibrium is dynamically inefficient. If $k^y<k^{y*}$, or alternatively $f'_y(k^y)=[u'(c_1)/\beta u'(c_2)(1-\tau)]>1$, then the economy is underinvesting relative to the golden rule and the steady state equilibrium is dynamically efficient.

V. LOCAL STABILITY AND DYNAMICS

The simple two-sector life-cycle structure above generates nonlinear maps. These nonlinear maps may generate cyclical behavior which is structurally stable, i.e. the qualitative behavior is preserved under small perturbations. The available tool to analyze these fluctuations is the theory of bifurcations. This is the theme of the analysis below.

V.1. Autarky

The dynamic equilibrium in autarky(A) is given by the system (9a)-(9b).

V.1.1. Logarithmic Utility Function

The case of logarithmic utility function is the case where $\partial s/\partial r_{t+1}{}^{j}=0$. If $\tau\neq 0$, this implies a saving function of the form $s_t{}^A=s(.)=S(p_t{}^A,p_{t+1}{}^A;\tau,\beta)$, i.e. $\partial S/\partial p_{t+1}{}^A\neq 0$. However, in the case where $\tau=0$, or alternatively no taxes on interest income are present, the saving function reduces to

$$s_t{}^A=s(.)=S(p_t{}^A;\beta)=[\beta/(1+\beta)]w_t,$$

where $\partial S/\partial p_t{}^A=[\beta/(1+\beta)](\partial w/\partial p_t{}^A)$ and $\partial S/\partial\beta=[w_t/(1+\beta)^2]>0$. If $k^x>k^y$, then $\partial S/\partial p_t{}^j<0$. If $k^x<k^y$, then $\partial S/\partial p_t{}^j>0$. It is analytically convenient to focus on this latter case($\tau=0$) for the moment.

The autarky equilibrium may be characterized by

$$k_{t+1}{}^A = y(p_t{}^A,k_t{}^A) \tag{12a}$$

$$S(p^A{}_t;\beta) = y(p_t{}^A,k_t{}^A). \tag{12b}$$

Equation (12b) implies $p^A{}_t=\phi(k^A{}_t;\beta)$ where

$\partial\phi/\partial k_t{}^A = (\partial y/\partial k_t{}^A)/[(\partial S/\partial p_t{}^A)-(\partial y/\partial p_t{}^A)]$,

$\partial\phi/\partial\beta = -(\partial S/\partial\beta)/[(\partial S/\partial p_t{}^A)-(\partial y/\partial p_t{}^A)]$.

This may be substituted into (12a) yielding $k_{t+1}{}^A=y[\phi(k^A{}_t;\beta),k_t{}^A]$ which implies that the dynamic equilibrium may be characterized by a single first order difference equation

$$k_{t+1}{}^A = \Psi(k^A{}_t;\beta) \tag{13}$$

where β is a parameter, k_o is given, and

$\partial\Psi(k^A{}_t;\beta)/\partial k_t{}^A = [(\partial y/\partial p_t{}^A)(\partial\phi/\partial k_t{}^A)+(\partial y/\partial k_t{}^A)]$,

$\partial\Psi(k^A{}_t;\beta)/\partial\beta = (\partial y/\partial p_t{}^A)(\partial\phi/\partial\beta)$.

If $k^x<k^y$, $\partial S/\partial p_t^A>0$ and $[(\partial S/\partial p_t^A)-(\partial y/\partial p_t^A)]>0$ unambiguously implying that $\partial\phi/\partial k_t^A<0$ and $\partial\phi/\partial\beta<0$. In this case, $\partial\Psi(k^A{}_t;\beta)/\partial k_t^A>0$ and $\partial\Psi(k^A{}_t;\beta)/\partial\beta>0$. If $k^x>k^y$, $\partial S/\partial p_t^A<0$ and $[(\partial S/\partial p_t^A)-(\partial y/\partial p_t^A)]$ could be either positive or negative. In the case when $[(\partial S/\partial p_t^A)-(\partial y/\partial p_t^A)]>0$, then $\partial\phi/\partial k_t^A<0$, $\partial\phi/\partial\beta<0$, $\partial\Psi(k^A{}_t;\beta)/\partial k_t^A$ is ambiguous, and $\partial\Psi(k^A{}_t;\beta)/\partial\beta<0$. In the possible case that $[(\partial S/\partial p_t^A)-(\partial y/\partial p_t^A)]<0$, then $\partial\phi/\partial k_t^A>0$, $\partial\phi/\partial\beta>0$, $\partial\Psi(k^A{}_t;\beta)/\partial k_t^A<0$, and $\partial\Psi(k^A{}_t;\beta)/\partial\beta<0$. The dynamic stability of the economy depends on the root $\partial\Psi(k^A;\beta)/\partial k_t^A$. The following propositions are in order:[9]

<u>Proposition 4</u>: If $\partial\Psi(k^A;\beta)/\partial k_t^A=[(\partial y/\partial p_t^A)(\partial\phi/\partial k_t^A)+(\partial y/\partial k_t^A)]>0$, then the autarky economy will exhibit a pitchfork bifurcation if there exists a $0<\beta_o<1$ such that: (i) $\Psi(0;\beta) = 0$ for all β in a small interval about β_o; (ii) $\partial\Psi(0;\beta_o)/\partial k_t^A = 1$; (iii) $\partial^2\Psi(0;\beta_o)/\partial k^2{}_t{}^A = 0$; (iv) $\partial^3\Psi(0;\beta_o)/\partial k^3{}_t{}^A \neq 0$; (v) $\partial^2\Psi(0;\beta_o)/\partial k_t^A\partial\beta_o \neq 0$.

In Proposition 4, conditions (i), (ii), and (iii) guarantee that β_o is a point of bifurcation of the dynamic equation (13). Conditions (iv) and (v) determine the direction and stability properties of the bifurcation. Specifically, condition (v) guarantees that $\partial\Psi(0;\beta_o)/\partial k_t^A$ crosses the unit circle when $\beta=\beta_o$ implying the bifurcation at $\beta=\beta_o$. It can be shown that, independently of relative factor intensities, $\partial^2\Psi(0;\beta_o)/\partial k_t^A\partial\beta_o>0$, furthermore

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9 *No explicit proofs of Propositions 4 and 5 (5′) are provided. The reader is referred to any text on bifurcation theory, e.g. Iooss(1979), Guckenheimer and Holmes(1983), Ruelle(1989). The general conditions for the bifurcations in the propositions below follow Grandmont(1988). It is assumed that the functions which underlie the equilibrium system are at least C^3, or at least three times differentiable. Across Propositions 4-5(5′) and Figures 1-3 a change of variable implies a rescaling of some price and quantity equilibrium to the origin.*

if one assumes that $\partial^3\Psi(0;\beta_o)/\partial k^3{}_t{}^A>0$, then Figure 1 illustrates the subcritical pitchfork bifurcation. In the region where $\beta>\beta_o$, the dynamics in the neighborhood of the origin are of unstable nature. At $\beta=\beta_o$, the origin is unstable and the root of the system is equal to one. In the region where $\beta<\beta_o$, multiple equilibria arises: one possible steady state equilibrium is $k^{A,1}$ which is unstable; the other possible steady state equilibrium is $k^{A,2}$ which coincides with the origin and is stable; the third possible steady state equilibrium is $k^{A,3}$ which is unstable.

<u>Proposition 5</u>: If $\partial\Psi(k^A;\beta)/\partial k_t{}^A=[(\partial y/\partial p_t{}^A)(\partial\phi/\partial k_t{}^A)+(\partial y/\partial k_t{}^A)]<0$, then the autarky economy will exhibit a flip bifurcation if there exists a $0<\beta_o<1$ such that: (i) $\Psi(0;\beta) = 0$ for all β in a small interval about β_o; (ii) $\partial\Psi(0;\beta_o)/\partial k_t{}^A = -1$; (iii) $\partial^2\Psi(0;\beta_o)/\partial k_t{}^A\partial\beta_o \neq 0$; (iv) $\partial^3\Psi^2(0;\beta_o)/\partial k^3{}_t{}^A \neq 0$.

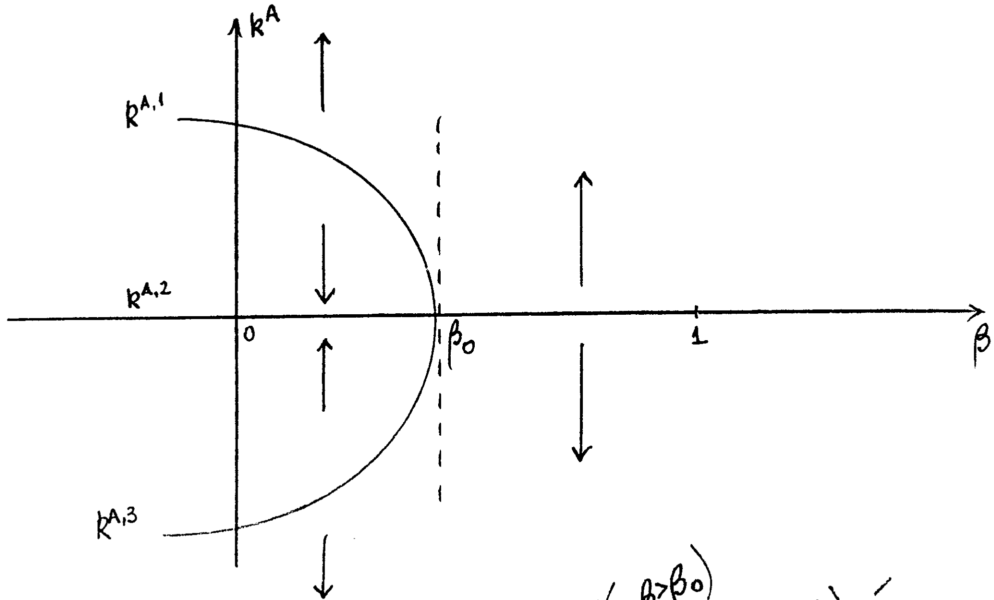
Condition (iv) in Proposition 5 allows one to discover a periodic orbit, of period two say, of the map $\Psi(.)$, where $\Psi^2(.)$ denotes a composition of the function $\Psi(.)$ with itself, or $\Psi^2(.;\beta)=\Psi(\Psi(.;\beta);\beta)$. An alternative way to formulate condition (iv) is:

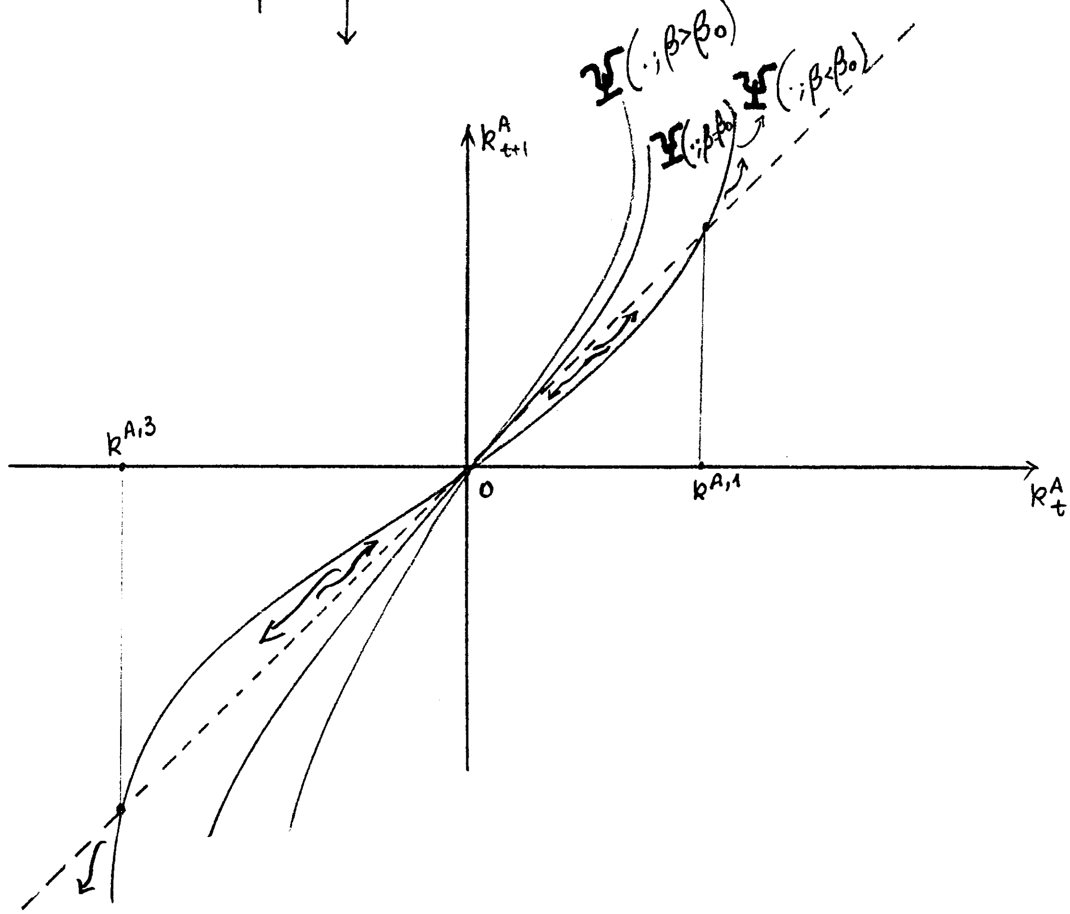
(iv') Sch $\Psi(0;\beta_o) = \{\{[\partial^3\Psi(0;\beta_o)/\partial k^3{}_t{}^A] / [\partial\Psi(0;\beta_o)/\partial k_t{}^A]\} - \{(3/2) ([\partial^2\Psi(0;\beta_o)/\partial k^2{}_t{}^A] / [\partial\Psi(0;\beta_o) / \partial k_t{}^A])^2\}\} \neq 0$;

where the operator Sch denotes the Schwarzian derivative. Again independently of relative factor intensities, $\partial^2\Psi(0;\beta_o)/\partial k_t{}^A\partial\beta_o>0$ and, if one assumes, in this case, that $\partial^3\Psi^2(0;\beta_o)/\partial k^3{}_t{}^A<0$, or alternatively $\mathrm{Sch}\Psi(0;\beta_o)<0$, a supercritical flip(period-doubling) bifurcation illustrated in Figure 2 obtains. Specifically, for $\beta>\beta_o$ there is no cycle of period two near $k^A=0$, and $k^A=0$ is a stable fixed point. However, for $\beta\leq\beta_o$, the point $k^A=0$ becomes

FIGURE 1.

SUBCRITICAL PITCHFORK BIFURCATION





unstable and a cycle of period two emerges. The stability of the period two cycle is guaranteed if the composite function $\Psi^2(.)$ satisfies $|\partial\Psi^2(.)/\partial k_t^A|<1$.

A qualification of the results above is in order. First, it is sharply established that countries with identical discount factors may be characterized by very distinct capital stocks in the long run. Indeed, the determinant of the capital stock in the long run will be the initial condition. Second, the examples above show the existence of multiple steady state equilibria and cyclical behavior associated with low discount factors, or alternatively high rates of time preference. This second remark is consistent with the results previously obtained by Benhabib and Nishimura(1985), Boldrin and Montrucchio(1986), Boldrin(1989), all using an infinitely lived optimal growth model. Finally, the cyclical behavior implied by Proposition 5 regards endogenous business cycles at the low frequency, as opposed to the high frequency real business cycles models as in Baxter(1988, 1992). This is because in this framework, each generation may live for an average of thirty years say.

V.1.2. $\partial s/\partial r_{t+1}^j>0$, or First and Second Period Consumption are Gross Substitutes

In this case, the savings function is $s_t^A=s(.)=S(p_t^A,p_{t+1}^A;\tau,\beta)$. The dynamic equilibrium in autarky is given by the system (9a)-(9b) which implies a pair of nonlinear difference equations for k and p given by

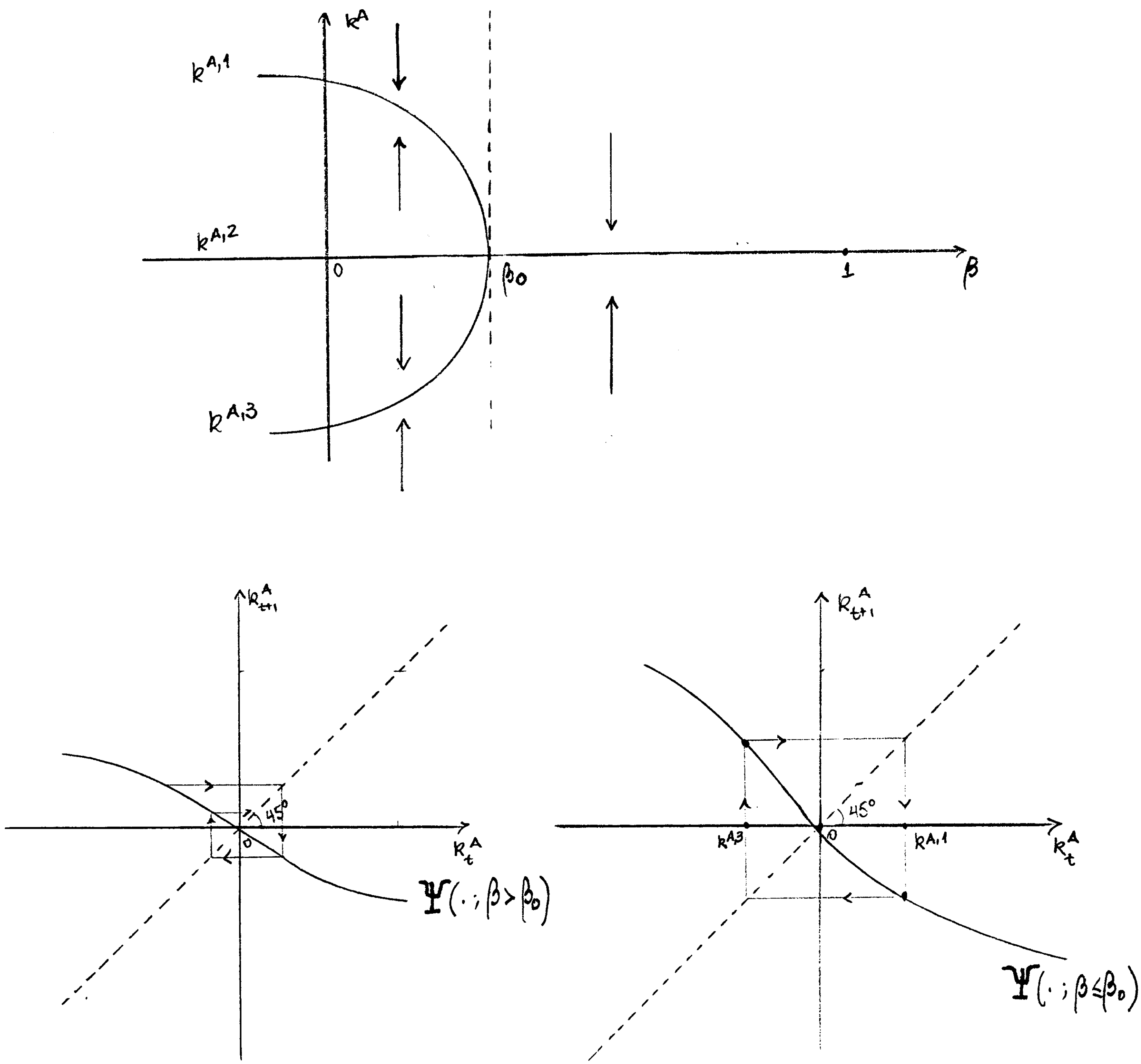
$$k_{t+1}^A - k_t^A = y(p_t^A,k_t^A) - k_t^A \tag{14a}$$

$$p_{t+1}^A - p_t^A = \psi(p_t^A,k_t^A;\tau,\beta) - p_t^A \tag{14b}$$

where (14b) is implicitly obtained from (9b), with

FIGURE 2

SUPERCRITICAL FLIP (PERIOD - DOUBLING) BIFURCATION



$\partial\psi/\partial p_t{}^A = [(\partial y/\partial p_t{}^A) - (\partial S/\partial p_t{}^A)]/(\partial S/\partial p_{t+1}{}^A);$

$\partial\psi/\partial k_t{}^A = (\partial y/\partial k_t{}^A)/(\partial S/\partial p_{t+1}{}^A);$

$\partial\psi/\partial\tau = -(\partial S/\partial\tau)/(\partial S/\partial p_{t+1}{}^A);$

$\partial\psi/\partial\beta = -(\partial S/\partial\beta)/(\partial S/\partial p_{t+1}{}^A).$

The steady state is characterized by a fixed point, $\{p^A, k^A\}$, which satisfies

$$k^A = y(p^A, k^A) \tag{15a}$$

$$p^A = \psi(p^A, k^A; \tau, \beta). \tag{15b}$$

The steady state equilibrium loci in p and k space are

$$dp^A/dk^A|_{kk} = \{1-[\partial y(p^A,k^A)/\partial k^A]\}/[\partial y(p^A,k^A)/\partial p_t{}^A] \tag{16a}$$

$$dp^A/dk^A|_{pp} = (\{[\partial\psi(p^A,k^A;\tau,\beta)/\partial k^A]/\{1-[\partial\psi(p^A,k^A;\tau,\beta)/\partial p_t{}^A]\}) + (1/\{1-[\partial\psi(p^A,k^A;\tau,\beta)/\partial p_t{}^A]\})[(\partial\psi(p^A,k^A;\tau,\beta)/\partial\tau)(d\tau/dk) + (\partial\psi(p^A,k^A;\tau,\beta)/\partial\beta)(d\beta/dk)]. \tag{16b}$$

The stability of the dynamic system (14) depends on the roots of[10]

$$\lambda_A{}^2 - (\text{tr}J)\,\lambda_A + \det J = 0 \tag{17}$$

where J is the Jacobian matrix of (14) evaluated at the steady state, and

$\text{tr}J = [\partial y(p^A,k^A)/\partial p_t{}^A] + [\partial\psi(p^A,k^A;\tau,\beta)/\partial k_t{}^A];$

$\det J = [\partial y(p^A,k^A)/\partial p_t{}^A]\ [\partial\psi(p^A,k^A;\tau,\beta)/\partial k_t{}^A] - [\partial y(p^A,k^A)/\partial k_t{}^A] \times [\partial\psi(p^A,k^A;\tau,\beta)/\partial p_t{}^A].$

The usual condition for stability of (14) is that $|\lambda_{Ai}|<1$ $(i=1,2)$ in (17). Galor(1992) examines the dynamic system (14a,b), from a local and global perspective, in great depth. My interest here is to examine the local behavior of the dynamic system, specifically the behavior of the roots of (17). Note that λ_{Ai} is an implicit function of τ and β, $\lambda_{Ai}(\tau,\beta)$. In

10 See e.g. Stokey, et. al.(1989), chapter 6. Burmeister, et. al.(1973) study the dynamic properties of a continuous time version of a multi-sector growth model, especially the convergence to the stable manifold and its relation to alternative savings assumptions.

particular, for a fixed $0<\beta<1$ there may exist a $0<\tau_o<1$ such that $\lambda_{Ai}(\tau_o)$ crosses the unit circle. In this case, the economy in question undergoes a bifurcation. The roots of (17) are

$$\lambda_{A1,2} = [trJ \pm \{([\partial y(p^A,k^A)/\partial k_t{}^A]-[\partial\psi(p^A,k^A;\tau,\beta)/\partial p_t{}^A])^2 + 4[\partial y(p^A,k^A)/\partial p_t{}^A][\partial\psi(p^A,k^A;\tau,\beta)/\partial k_t{}^A]\}^{1/2}]/2. \tag{18}$$

Case 1: $k^x < k^y$ or the capital good is capital intensive

It is possible to show that in this case the characteristic equation (17) presents no complex roots and the Hopf bifurcation is ruled out.[11] Furthermore, in this case, trJ>0 and detJ>0 implying that the two roots are strictly positive, no cyclical behavior occurs in this economy. It can be shown that at least one of the roots is greater than one, and that the steady state is *locally* either a saddle point (if the other root is less than one) or totally unstable (if the other root is grater than one). The following situation may emerge: if $\lambda_{A1}(.)>1$, then $\lambda_{A2}(\tau_o)$ crosses the unit circle at $\tau=\tau_o$. One possibility is illustrated in Figure 3. The pitchfork bifurcation point is at the origin where $p^A=k^A=0$. For $\tau<\tau_o$, $\lambda_{A2}(.)<1$ and $0<[dp^A/dk^A|_{kk}]<[dp^A/dk^A|_{pp}]$ in the neighborhood of $p^A=k^A=0$, or alternatively the kk locus is less steep than the pp locus in the neighborhood around the origin and the origin is a locally unique saddle point equilibrium. At $\tau=\tau_o$, $\lambda_{A2}(.)$ crosses the unit circle and the manifold associated with that root is neutral. For $\tau>\tau_o$, $\lambda_{A2}(.)>1$ and $[dp^A/dk^A|_{kk}]>[dp^A/dk^A|_{pp}]>0$ in the neighborhood of $p^A=k^A=0$, or alternatively the kk locus is more steep than the

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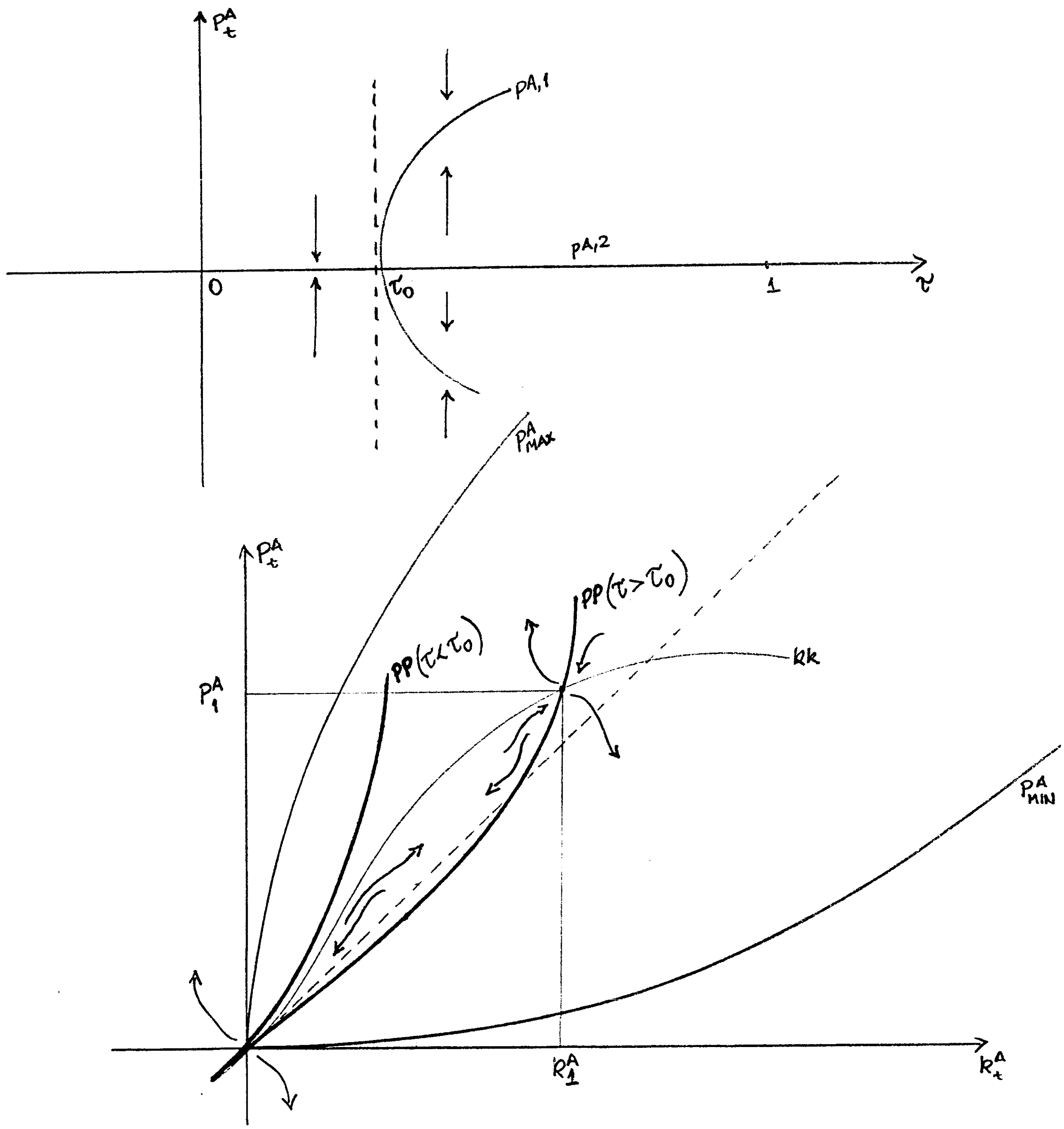
11 Farmer(1986) analyzes conditions under which the Hopf bifurcation emerges in a Diamond(1965) like model.

FIGURE 3

$\partial s/\partial r_{t+1} > 0$ AND $R^x < R^y$



NOTE: P^A_{MAX} AND P^A_{MIN} ARE FUNCTIONS OF k^A AND THEY DETERMINE THE BOUNDS WITHIN BOTH GOODS ARE PRODUCED. THE FIGURE IN THE 3RD QUADRANT IS SYMMETRIC TO THE FIRST QUADRANT.

pp locus in the neighborhood around the origin. The origin becomes totally unstable and other equilibria emerge, $\{p^{A,1},k^{A,1}\}$ say which is a saddle point.

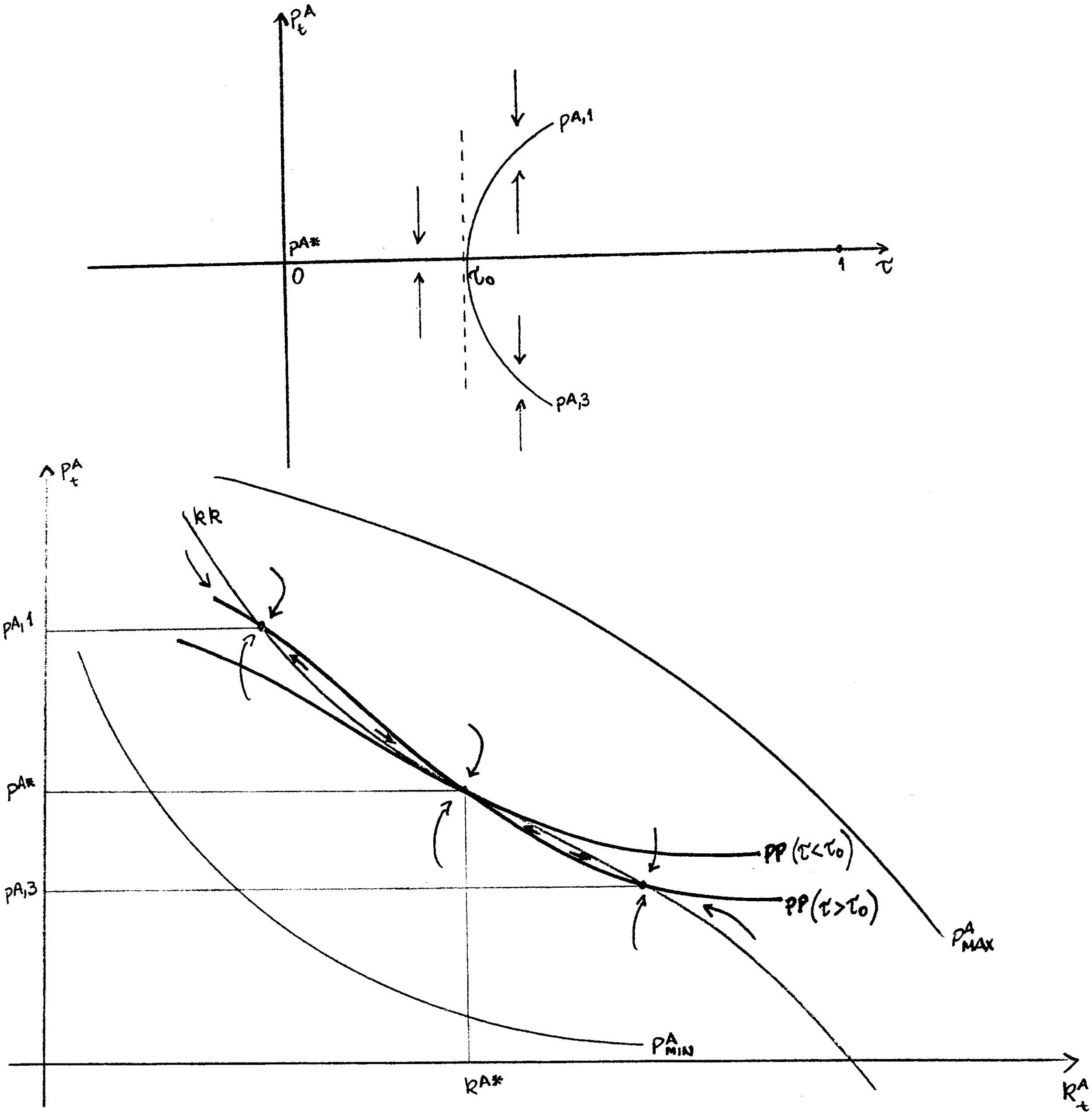
Case 2: $k^x > k^y$ or the consumption good is capital intensive

Again, the characteristic equation (17) presents no complex roots. However, in this case, cyclical behavior may occur. It can be shown that one of the roots is unambiguously less than zero, $\lambda_{A2}(.)<0$ say. The other root may be either positive or negative implying that a variety of cases are possible. The following case is possible: if $-1<\lambda_{A2}(.)<0$, then $\lambda_{A1}(\tau_o)>0$ crosses the unit circle at $\tau=\tau_o$. One possibility is illustrated in Figure 4. The bifurcation point is at the point $\{p^{A*},k^{A*}\}$. For $\tau<\tau_o$, $\lambda_{A1}(.)<1$ and $[dp^A/dk^A|_{kk}]<[dp^A/dk^A|_{pp}]<0$ in the neighborhood of $\{p^{A*},k^{A*}\}$, or alternatively the kk locus is less steep than the pp locus in the neighborhood around $\{p^{A*},k^{A*}\}$ and $\{p^{A*},k^{A*}\}$ is a locally unique stable equilibrium. At $\tau=\tau_o$, $\lambda_{A1}(.)$ crosses the unit circle and the manifold associated with that root is neutral. For $\tau>\tau_o$, $\lambda_{A1}(.)>1$ and $0>[dp^A/dk^A|_{kk}]>[dp^A/dk^A|_{pp}]$ in the neighborhood of $\{p^{A*},k^{A*}\}$, or alternatively the kk locus is more steep than the pp locus in the neighborhood around $\{p^{A*},k^{A*}\}$. The point $\{p^{A*},k^{A*}\}$ becomes a saddle point and other equilibria emerge, $\{p^{A,1},k^{A,1}\}$ say which is stable. A more interesting possibility arises when one of the roots crosses minus one (at some $\tau=\tau_o$) because a cycle, of period two say, emerges. The Appendix shows that using the Center Manifold Reduction Theorem, the behavior of solutions to (14a,b) near the steady state $\{p^{A*},k^{A*}\}$ amounts to analyzing the local behavior of the single first order nonlinear difference equation

$$p_{t+1}{}^A = \psi[p_t{}^A,\ \Phi(p_t{}^A) + O(|p_t{}^A-p^{A**}|^q);\tau,\beta] \tag{19}$$

FIGURE 4

$\partial s/\partial r_{t+1} > 0$ AND $k^x > k^y$



for some $q>1$. The term $0(|p_t^A - p^{A**}|^q)$ denotes the order of the approximation(see the Appendix) of the function $\Phi(p_t^A)$ to k_t^A.[12] The analogous to Proposition 5 for the map (19) is

<u>Proposition 5'</u>: The autarky economy will exhibit a flip bifurcation if, given $0<\beta<1$ and $\partial s/\partial r_{t+1}>0$, there exists a $0<\tau_o<1$ such that for some $q>1$: (i) $\psi[p^{A*},\Phi(p^{A*})+0(|p^{A*}-p^{A**}|^q);\tau,\beta] = p^{A*}$ for all τ in a small interval about τ_o; (ii) $\partial\psi[p^{A*},\Phi(p^{A*})+0(|p^{A*}-p^{A**}|^q);\tau_o,\beta]/\partial p_t^A = -1$; (iii) $\partial^2\psi[p^{A*},\Phi(p^{A*})+0(|p^{A*}-p^{A**}|^q);\tau_o,\beta]/\partial p_t^A\partial\tau_o \neq 0$; (iv) $\mathrm{Sch}\psi[p^{A*},\Phi(p^{A*})+0(|p^{A*}-p^{A**}|^q);\tau_o,\beta] \neq 0$.

Under the appropriate signs for (iii) and (iv), period two cycles illustrated in Figure 2 emerge for the capital stock and the relative price in this economy.

As opposed to Grandmont(1985) and Benhabib and Laroque(1988), in the two-sector model the emergence of cycles does not depend on the assumption that $\partial s/\partial r_{t+1}^j<0$. Reichlin(1986) also shows that this result obtains in a one-sector productive overlapping generations model due to a production parameter. However, as shown in the example economy above, cyclical behavior is, among other things, a function of $\partial s/\partial r_{t+1}^j$. In addition, the important result that is established here, for $\partial s/\partial r_{t+1}^j\geq 0$, regards the relative intensity of capital and labor in the production of both goods: *for cyclical behavior to occur the assumption that the consumption good be capital intensive is necessary (but not sufficient).* The result above is shared with

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12 *Note that all the analysis of this subsection may be performed using the map (19).*

Benhabib and Nishimura(1985), Reichlin(1987), and Kalra(1990) each under a specific set of different assumptions on saving behavior, technologies, and so on. Intuitively, cyclical behavior emerges because the higher saving induces a higher capital stock, but the gains are borne by the sector that produces the consumption good, not the sector that produces the capital good, leading to oscillations in accumulation.[13]

The example economies above clearly show that two countries with identical "high" tax rates may be characterized by distinct long run capital stocks. In this sense, tax policy, from the point of view of Parente and Prescott(1991), does not explain long run differences in the capital stock across countries. In fact, it is the initial capital stock that shall endogenously determine the size of the long run capital stock. This result is shared with Lucas(1988), where this phenomenon is associated with the state of knowledge embodied in human capital, and with Boldrin and Scheinkman(1988) where a parameter of learning-by-doing is the main determinant.

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13 Note that the capital intensity condition that guarantees cyclical behavior through bifurcation theory is the same condition that guarantees uniqueness and stability of the balanced capital/labor ratio in Uzawa(1961), page 45, and Uzawa(1963), page 111-112. However, in Uzawa(1963), section 6, he reverts his previous results by assuming inflexibity of factor prices and involuntary unemployment. In my model, what accounts for such opposite result is indeed the saving behavior induced by the overlapping generations model with heterogeneous agents; the prices of factors are fully flexible and factors are fully employed.

V.2. Small Open Economy

Assume that the economy is small, in the sense that it imports and exports both goods at given world prices. In turn, ${p_t}^S = p^S$ constant, which implies that ${s_t}^S = S(p^S;\tau,\beta) = s^S$ constant. It implies, by (9), that

$$k_{t+1}{}^S = S(p^S;\tau,\beta) = y({k_t}^S;p^S) \qquad (20)$$

where p^S is now a parameter. Clearly, this economy presents no dynamics, and there is a unique fixed point such that $k^S = S(p^S;\tau,\beta) = y(k^S;p^S)$, or alternatively no possibility of multiple steady state equilibria arises.[14]

V.3. Two-Country World

Equations (10a)-(10b) may be substituted into (10c) to represent the dynamic trade equilibrium by a second order nonlinear difference equation in the relative price given by

$$p_{t+1} = H(p_t, p_{t-1};\tau^D,\tau^F,\beta^D,\beta^F) \qquad (21)$$

where

$\partial H/\partial p_t = \{2[(\partial y/\partial p_t) - (\partial S/\partial w)(\partial w/\partial p_t) - (\partial S/\partial p_t)] + \{[(\partial y/\partial k^D{}_t) + (\partial y/\partial k^F{}_t)]$ $[(\partial S/\partial r)(\partial r/\partial p_t) + (\partial S/\partial p_t)]\}\} / \{2[(\partial S/\partial r)(\partial r/\partial p_{t+1}) + (\partial S/\partial p_{t+1})]\};$

$\partial H/\partial p_{t-1} = \{[(\partial y/\partial k^D{}_t) + (\partial y/\partial k^F{}_t)]\ [(\partial S/\partial w)(\partial w/\partial p_{t-1}) + (\partial S/\partial p_{t-1})]\} /$ $\{2[(\partial S/\partial r)(\partial r/\partial p_{t+1}) + (\partial S/\partial p_{t+1})]\};$

$\partial H/\partial \tau^D = \{(\partial S/\partial \tau^D)\ [(\partial y/\partial k^D{}_t) - 1]\} / \{2[(\partial S/\partial r)(\partial r/\partial p_{t+1}) + (\partial S/\partial p_{t+1})]\};$

$\partial H/\partial \tau^F = \{(\partial S/\partial \tau^F)\ [(\partial y/\partial k^F{}_t) - 1]\} / \{2[(\partial S/\partial r)(\partial r/\partial p_{t+1}) + (\partial S/\partial p_{t+1})]\};$

---

14 Kalra(1990) analyzes a small open economy where there is a consumption good and a "quasi" consumption-"quasi" capital good. In his system, the terms of trade defined as the price of imported capital to domestic exportables is a parameter and he obtains multiple equilibria and dynamic cycles in this special case.

$\partial H/\partial\beta^D = \{(\partial S/\partial\beta^D)\ [(\partial y/\partial k^D{}_t) - 1]\} / \{2[(\partial S/\partial r)(\partial r/\partial p_{t+1}) + (\partial S/\partial p_{t+1})]\};$

$\partial H/\partial\beta^F = \{(\partial S/\partial\beta^F)\ [(\partial y/\partial k^F{}_t) - 1]\} / \{2[(\partial S/\partial r)(\partial r/\partial p_{t+1}) + (\partial S/\partial p_{t+1})]\}.$

The dynamic stability of the world economy depends on the roots of the characteristic equation

$$\lambda^2 - [\partial H(p,p;\tau^D,\tau^F,\beta^D,\beta^F)/\partial p_t]\lambda - [\partial H(p,p;\tau^D,\tau^F,\beta^D,\beta^F)/\partial p_{t-1}] = 0 \quad (21)$$

where the steady state is given by (11a,b,c). A sufficient, but not necessary, condition to rule out complex roots in (21) is that $\partial H(p,p;\tau^D,\tau^F,\beta^D,\beta^F)/\partial p_{t-1}>0$. Under this condition, the two roots are of opposite sign $[\lambda_1\times\lambda_2=-\partial H(p,p;\tau^D,\tau^F,\beta^D,\beta^F)/\partial p_{t-1}<0]$. A variety of cases are possible depending on the relative factor intensities and the interest sensitivity of the savings function and the analysis may be carried out in the same fashion as in sections V.1.1. and V.1.2. above. Clearly, for a fixed $0<\beta^j<1$, for j equal D and F, there may exist a $0<\tau_o{}^j<1$, for j either D or F, such that $\lambda_1(\tau_o{}^j) = \pm 1$, and Propositions 4, 5, and 5′ may emerge for the world economy. Multiplicity of equilibria and cyclical behavior are likely to occur in a two-country two-sector overlapping generations world. Most importantly: *even if the two countries are identical in every respect, the analysis of sections V.1.1. and V.1.2. above apply, and the two countries may end up with distinct long run capital stocks determined by its initial condition.*

VI. CONCLUDING REMARKS

The main results of this paper are:

i. Propositions 1, 2, and 3 show that different tax rates (or discount factors) may explain the long run pattern of trade; in the life-cycle framework long run factor price equalization holds along with diversity of production; contrary to the case of specialization in the infinitely lived representative agent model of Baxter(1992)[and Stiglitz(1970)], the heterogeneity implied by the life-cycle framework allows for long run diversity of production;

ii. In the case of autarky and logarithmic utility function, if the tax rate is positive, saving is a function of the current and future relative price; however, if the tax rate is zero, saving is a function only of the current relative price;

iii. Multiple steady state equilibria and endogenous cycles may emerge in the autarky equilibrium with logarithmic utility function for a given low discount factor (or high tax rate); if the capital good is capital intensive multiple equilibria arises; if the consumption good is capital intensive cyclical behavior arises; the long run capital stock will be determined by its initial condition;

iv. Multiple steady state equilibria and endogenous cycles may emerge in the autarky equilibrium when first and second period consumption are gross substitutes for a given high tax rate (however, the Hopf bifurcation does not occur); if the capital good is capital intensive multiple equilibria arises; if the consumption good is capital intensive cyclical behavior arises; the long run capital stock will be determined by its initial condition;

v. The small open economy presents no dynamics and a unique long run equilibrium;

vi. The two-country world may present multiple equilibria and cyclical behavior for a given tax rate or discount factor; even if countries are equal in all respects, multiple equilibria is possible and disparities across countries are possible; the long run capital stock will be determined by its initial condition;

Future research includes an extension to government debt and international borrowing and lending.

APPENDIX

I. Proof of Proposition 1:

Without loss of generality, assume $\tau^D = \tau^F + d\tau$; $d\tau > 0$. From the steady state trade equilibrium, (11a,b,c), we have

$$S(.;\tau^D) < S(.;\tau^F)$$

which implies that $k^F = k^D + dk$; $dk > 0$. By (11c), it implies

$$y(.,k^D) + y(.,k^D + dk) = 2k^D + dk.$$

A first order Taylor's expansion of $y(.,k^D + dk)$ is

$$y(p,k^D) + [\partial y(p,k_1)/\partial k]\ dk$$

[where $k^D \le k_1 \le k^D + dk$] which upon substitution in the expression above yields

$$2y(p,k^D) + \{[\partial y(p,k_1)/\partial k] - 1\}\ dk - 2k^D = 0.$$

By the Rybczynsky Theorem, if $k^x > k^y$, then $\partial y/\partial k < 0$. Then, the expression above implies

$$y(p,k^D) - k^D > 0$$

or the high tax(domestic) country exports the labor intensive(capital) good.■

II. Proof of Proposition 2:

First, I must show that if $k^x > k^y$ and $\tau^D > \tau^F$, then $p^{A,F} < p < p^{A,D}$, or the relative price in trade equilibrium lies in between the autarky relative prices and the relative price in the domestic country is higher.

Without loss of generality, there exists some $dp^{A,D}$ and $dp^{A,F}$ such that

$$p = p^{A,D} + dp^{A,D} = p^{A,F} + dp^{A,F}.$$

Substituting the above into the trade equilibrium (11a,b,c) yields

$$k^D = S(p^{A,D} + dp^{A,D};\tau^D)$$

$$k^F = S(p^{A,F} + dp^{A,F};\tau^F)$$

$$S(p^{A,D}+dp^{A,D};\tau^D) + S(p^{A,F}+dp^{A,F};\tau^F) = y(p^{A,D}+dp^{A,D},k^D) + y(p^{A,F}+dp^{A,F},k^F).$$

A first order Taylor's expansion of S(.) is

$$S(p^{A,D};\tau^D) + [\partial S(p_1{}^D;\tau^D)/\partial p^D]\ dp^{A,D} = k^{A,D} + [\partial S(p_1{}^D;\tau^D)/\partial p^D]\ dp^{A,D}$$

where $p_1{}^D$ is an element of the set $[p^{A,D},\ p^{A,D} + dp^{A,D}]$ and the equality derives from the autarky saving and investment equilibrium. Similarly, for the foreign economy

$$S(p^{A,F};\tau^F) + [\partial S(p_1{}^F;\tau^F)/\partial p^F]\ dp^{A,F} = k^{A,F} + [\partial S(p_1{}^F;\tau^F)/\partial p^F]\ dp^{A,F}$$

where $p_1{}^F$ is an element of the set $[p^{A,F},\ p^{A,F} + dp^{A,F}]$ and the equality derives from the autarky saving and investment equilibrium.

A first order Taylor's expansion of y(.) is

$$y(p^{A,D},k^{A,D})+[\partial y(p_2{}^D,k_2{}^D)/\partial p^D]\ dp^{A,D}+[\partial y(p_2{}^D,k_2{}^D)/\partial k^D][\partial S(p_1{}^D;\tau^D)/\partial p^D]dp^{A,D}$$

where $p_2{}^D$ is an element of the set $[p^{A,D},\ p^{A,D} + dp^{A,D}]$ and $k_2{}^D$ is an element of the set $[k^{A,D},\ k^{A,D} + [\partial S(p_1{}^D;\tau^D)/\partial p^D]dp^{A,D}]$. Similarly, for the foreign economy

$$y(p^{A,F},k^{A,F})+[\partial y(p_2{}^F,k_2{}^F)/\partial p^F]\ dp^{A,F}+[\partial y(p_2{}^F,k_2{}^F)/\partial k^F][\partial S(p_1{}^F;\tau^F)/\partial p^F]dp^{A,F}$$

where $p_2{}^F$ is an element of the set $[p^{A,F},\ p^{A,F} + dp^{A,F}]$ and $k_2{}^F$ is an element of the set $[k^{A,F},\ k^{A,F} + [\partial S(p_1{}^F;\tau^F)/\partial p^F]dp^{A,F}]$.

The steady state world capital good market (dis)equilibrium is then

$$y(p,k^D) + y(p,k^F) - S(p,p;\tau^D) - S(p,p;\tau^F) = y(p^{A,D},k^{A,D}) + [\partial y(p_2{}^D,k_2{}^D)/\partial p^D]$$
$$dp^{A,D} + [\partial y(p_2{}^D,k_2{}^D)/\partial k^D][\partial S(p_1{}^D;\tau^D)/\partial p^D]dp^{A,D} + y(p^{A,F},k^{A,F}) +$$
$$[\partial y(p_2{}^F,k_2{}^F)/\partial p^F]\ dp^{A,F} + [\partial y(p_2{}^F,k_2{}^F)/\partial k^F][\partial S(p_1{}^F;\tau^F)/\partial p^F]dp^{A,F} - k^{A,D} -$$
$$[\partial S(p_1{}^D;\tau^D)/\partial p^D]\ dp^{A,D} - k^{A,F} - [\partial S(p_1{}^F;\tau^F)/\partial p^F]\ dp^{A,F}.$$

Collecting terms and recognizing that the autarky equilibrium implies $y(p^{A,j},k^{A,j}) = k^{A,j}$, one obtains

$$y(p,k^D) + y(p,k^F) - S(p,p;\tau^D) - S(p,p;\tau^F) = dp^{A,D}\{[\partial y(p_2{}^D,k_2{}^D)/\partial p^D] +$$
$$[\partial S(p_1{}^D;\tau^D)/\partial p^D]([\partial y(p_2{}^D,k_2{}^D)/\partial k^D] - 1)\} + dp^{A,F}\{[\partial y(p_2{}^F,k_2{}^F)/\partial p^F] +$$
$$[\partial S(p_1{}^F;\tau^F)/\partial p^F]([\partial y(p_2{}^F,k_2{}^F)/\partial k^F] - 1)\}.$$

Assumptions i, ii, and iii of the proposition imply that the coefficients on $dp^{A,D}$ and $dp^{A,F}$ in the expression above have the same sign. Therefore, the long run trade equilibrium (11c) is satisfied if and only if $dp^{A,D}$ and $dp^{A,F}$ are of opposite sign. Since $\tau^D > \tau^F$ and $k^{x,j} > k^{y,j}$, it follows directly from Proposition 1 that $p^{A,D} > p^{A,F}$, and $k^{A,D} < k^{A,F}$. It must be the case that $p^{A,F} < p < p^{A,D}$, or alternatively $dp^{A,D} < 0$ and $dp^{A,F} > 0$.

The Stolper-Samuelson theorem implies that if $k^{x,j} > k^{y,j}$, then

$$dw^{A,j}/dp^{A,j} < 0 \quad \text{and} \quad dr^{A,j}/dp^{A,j} > 0.$$

Noting from above that $dp^{A,D} < 0$ and $dp^{A,F} > 0$, opening up trade implies: i. the wage rate decreases and the rental rate increases in the low tax(foreign) country; ii. the wage rate increases and the rental rate decreases in the high tax(domestic) country.∎

III. Proof of Proposition 3:

In the trade equilibrium, goods prices are equalized every period by (10a,b,c). If both goods are produced, by (4a,b), factor prices are equalized every period, as well as in the steady state trade equilibrium.∎

IV. The Center Manifold Reduction Theorem

The analysis below is based on Grandmont(1988) and Reichlin(1987), see also Iooss(1979) and Guckenheimer and Holmes(1983). The Center Manifold Reduction Theorem permits one to reduce the dimension of a given nonlinear dynamical system to the number of eigenvalues which cross the unit circle in the bifurcating family. In the system (14a,b) I am interested in the case where one eigenvalue crosses the unit circle. Therefore, a local center manifold may be represented by a function

$$k_t^A = \xi(p_t^A)$$

that satisfies the following properties:

i. $\xi\{\psi[p^{A*},\xi(p^{A*});\tau,\beta]\} = p^{A*}$

ii. $\psi[p^{A*},\xi(p^{A*});\tau_o,\beta] = p^{A*}$

iii. $\partial\psi[p^{A*},\xi(p^{A*});\tau,\beta]/\partial p_t^A = -1$.

The effort is concentrated in finding the function $\xi(.)$. Consider another function $\Phi(p_t^A)$ having a fixed point at p^{A**} and the same slope as $\xi(.)$ at p^{A**} or alternatively:

i. $\Phi(p^{A**}) = p^{A**}$

ii. $\Phi'(p^{A**}) = \{1-(\partial\psi[p^{A**},\xi(p^{A**});\tau,\beta]/\partial p_t^{A})\}/\{\partial\psi[p^{A**},\xi(p^{A**});\tau,\beta]/\partial k_t^{A}\}$.

Now, define

$$N(p^{A*}) = \Phi\{\psi[p^{A*},\Phi(p^{A*});\tau,\beta]\} - p^{A*}$$

and expand N(.) to obtain

$$N(p^{A*}) = 0(|p^{A*} - p^{A**}|^{q})$$

as p^{A*} approaches p^{A**}, for some q>1. The notation $0(|p^{A*} - p^{A**}|^{q})$ indicates the accuracy of the approximation. If q=1, the difference in $N(p^{A*})$ is bounded by a first order linear approximation; if q=2, it is bounded by a second order quadratic approximation, and so on. Then, [Grandmont(1988), page 36; Reichlin(1987), page 55; Guckenheimer and Holmes(1983), chapter 3, Iooss(1979), chapter 5]

$$\xi(p^{A*}) = \Phi(p^{A*}) + 0(|p^{A*} - p^{A**}|^{q})$$

as p^{A*} approaches p^{A**}, for some q>1. Equation (19) in the text is obtained by substituting the above expression into (14b).

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